1 Functions

- What is a function? A rule/machine.

$$x \to f(x) \to y$$

- Vertical line test (1 y for each x).

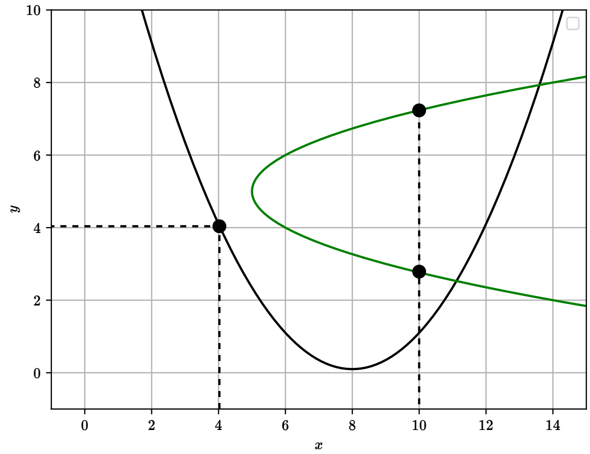


Figure 1: Vertical Line Test

1.1 Different kinds of functions

- Positive function: above x-axis.

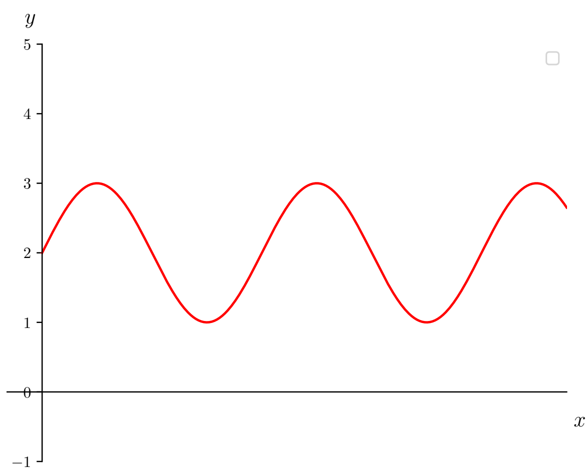


Figure 2: Positive Function

- Negative function: below x-axis.

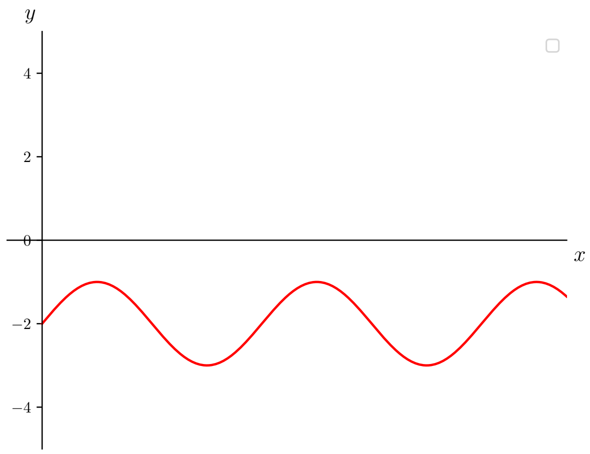


Figure 3: Negative Function

- Both: $y = \sin(x)$

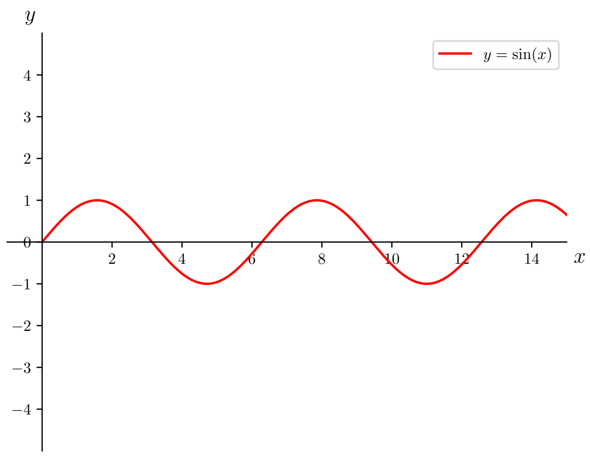


Figure 4: $y = \sin(x)$

- Strictly increasing: $y = e^x$

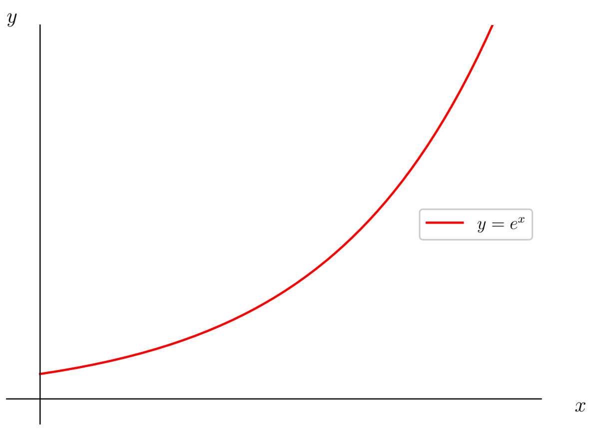


Figure 5: Strictly increasing function

- Monotonically decreasing function.

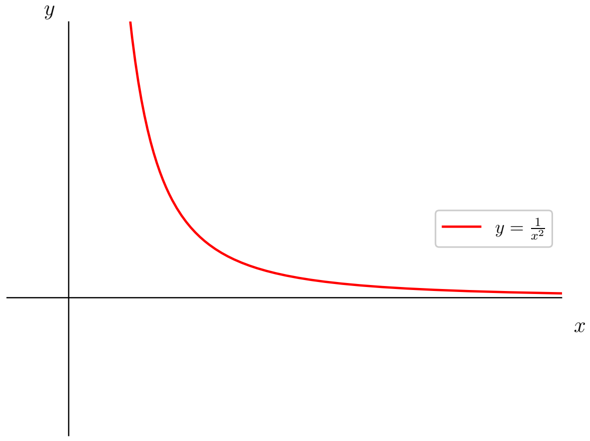


Figure 6: Monotonically decreasing function

- Constant function: $y = c$

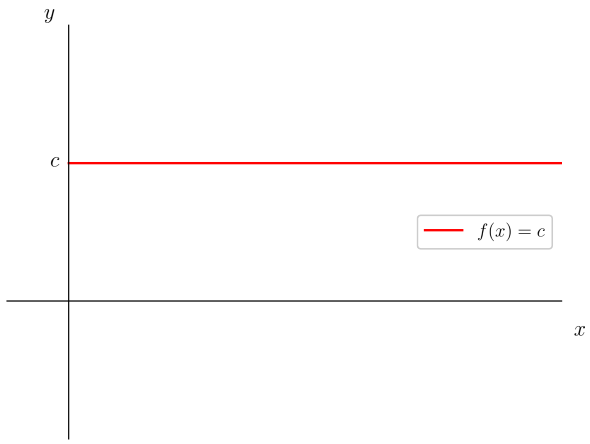


Figure 7: Constant function: $f(x) = c$

- Even function: Symmetric w.r.t y-axis. $f(-x) = f(x)$

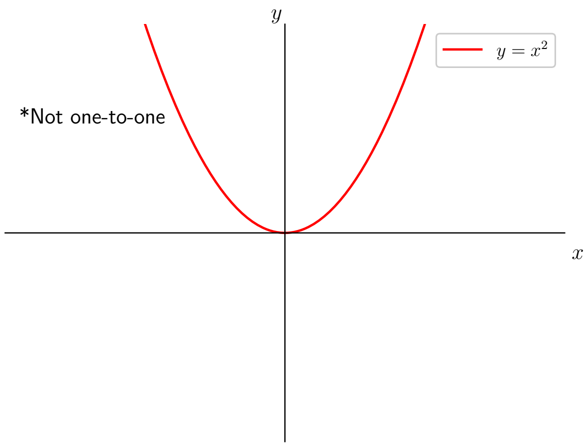


Figure 8: Even function: $y = x^2$

- Odd function: Symmetric wert the origin. $f(-x) = -f(x)$

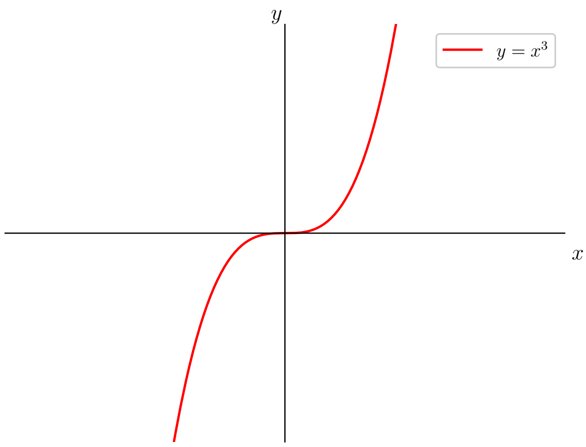


Figure 9: Odd function: $y = x^3$

- Step function:

$$f(x) = \begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

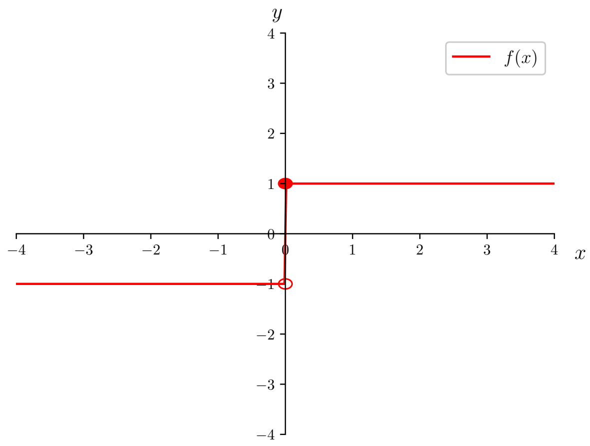


Figure 10: Step Function

- Greatest integer function. For example $y = [[x]]$

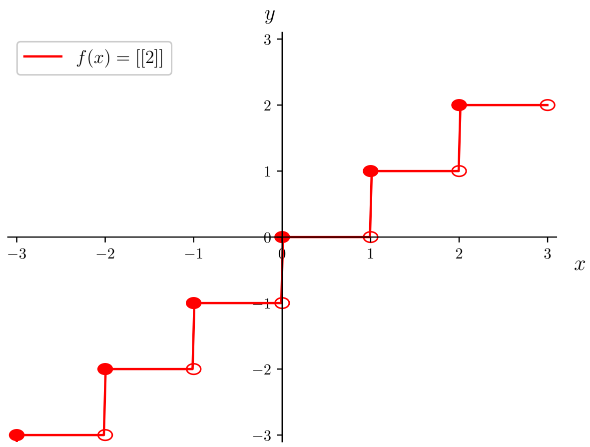


Figure 11: Greatest integer that is ≤ 2

- Continuous/Discontinuous Function

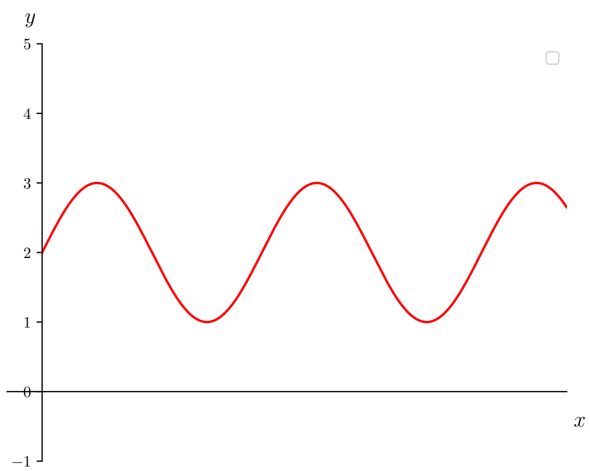


Figure 12: Continuous Function

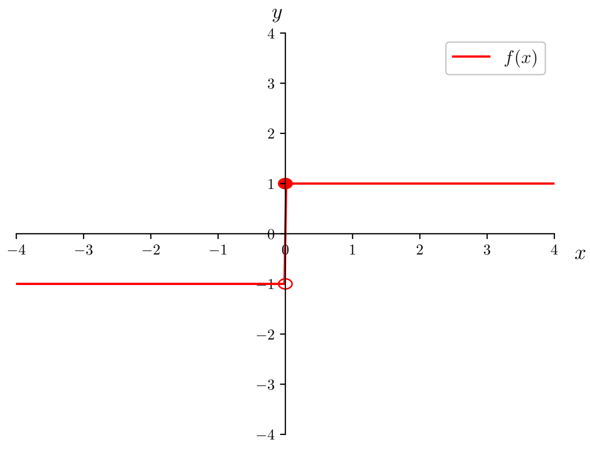


Figure 13: Discontinuous Function

- One-to-One function. For $x_1 \neq x_2, f(x_1) \neq f(x_2)$ horizontal line test

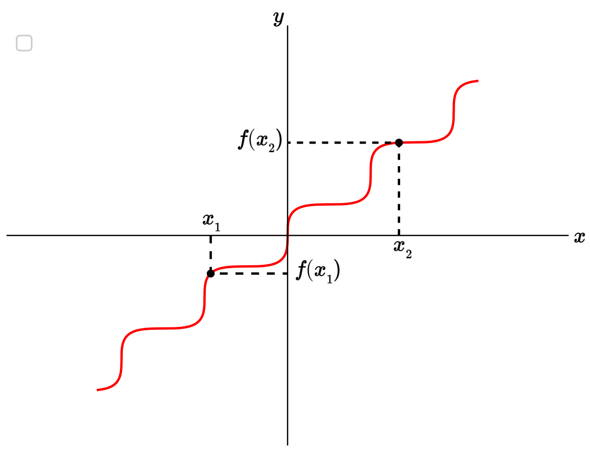
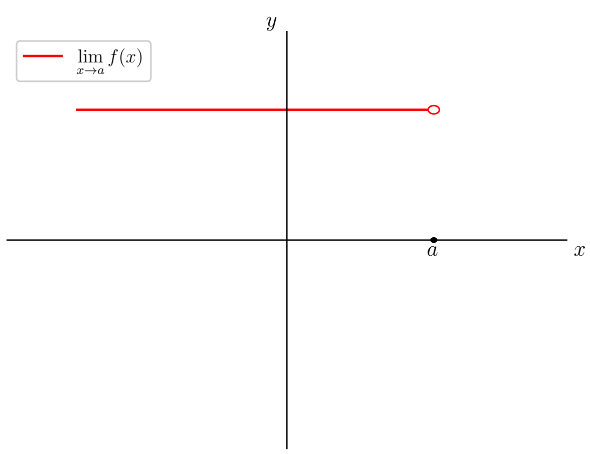


Figure 14: One-to-One Function

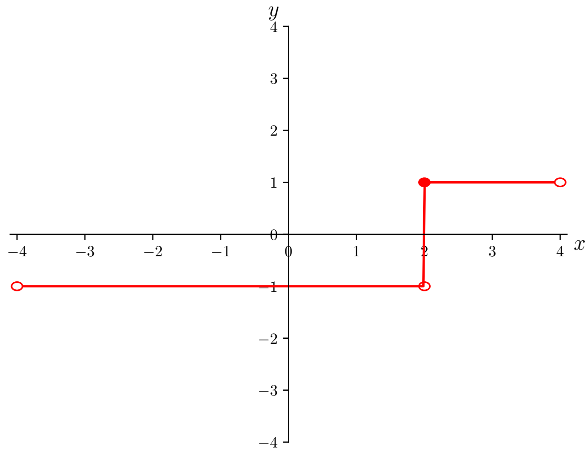
2 Calculus

Calculus 1 starts with limits.



Recall the step function:

$$f(x) = \begin{cases} -1 & x < 2 \\ 1 & x \geq 2 \end{cases}$$



$$\lim_{x \to 3} f(x) = 1$$

$$\lim_{x \to -1} f(x) = -1$$

$$\lim_{x \to 2} f(x) = \begin{cases} \lim_{x \to 2^+} f(x) = 1 \\ \lim_{x \to 2^-} f(x) = -1 \end{cases} \quad \therefore \text{ DNE}$$

where $x \to a^+$is read as 'x *approaches to* a *from right*', and $x \to \bar{a}$ is read as '*x approaches to* a *from left*'.

2.1 Limits

$$\lim_{x \to a} f(x) = L$$

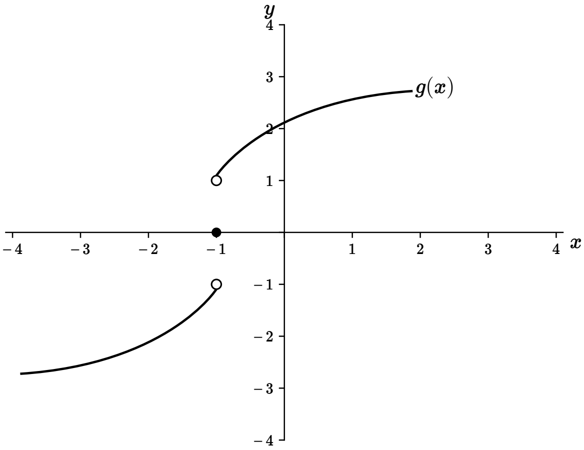
The limit exists and equals L.

1. Make a table of x and y values and check what the y-values approach to when x approaches 0. Consider the limit: $\lim_{x \to 0} \left(x^2 - x + 1\right)$:

x	-0.1	-0.01	0	0.01	0.1
$y = x^2 - x + 1$	$\rightarrow$	$\rightarrow$	1	$\leftarrow$	$\leftarrow$

$$\Rightarrow \lim_{x \to 0} \left(x^2 - x + 1\right) = 1$$

2. Use a graph.



$$\lim_{x \to -1^+} g(x) = 1$$
$$\lim_{x \to -1^-} g(x) = -1$$
$$\therefore \lim_{x \to 1} g(x) = \text{DNE}$$

3. Use the following rules:

   RULE 0: Plug in $\to$ number (this is the limit).

$$\lim_{x \to c} g(x) = g(c)$$

   For example,

$$\begin{aligned}\lim_{x \to \frac{\pi}{2}} \sin(\pi + 2x) &= \sin\left(\pi + 2\left(\frac{\pi}{2}\right)\right) \\ &= \sin(2\pi) \\ &= 0 \\ \lim_{x \to \frac{\pi}{2}} \sin(x) &= \sin\left(\frac{\pi}{2}\right) \\ &= 1\end{aligned}$$

   RULE 1: Plug in $\to$ no fixed value.

$$\lim_{x \to \infty} \sin(x) \neq \text{ fixed no. } \Rightarrow \text{ DNA}$$

   Techniques to find the limit in this case:

   a) Divide out
   b) Rationalize

   Tip

   Evil factor $= x - a$
   find $x - a$ and cancel it out!

a) Divide out

Example:

$$\lim_{x\to 0}\frac{x^2-4x}{x^2-3x-4}=\frac{0}{4}$$

Example:

$$\begin{aligned}
&\lim_{x\to 4}\frac{x^2-4x}{x^2-3x-4}=\frac{0}{0}\ \text{(undetermined form)}\\
\hookrightarrow &\lim_{x\to 4}\frac{x(x-4)}{(x-4)(x+1)}=\lim_{x\to 4}\frac{x}{x+1}=4/5
\end{aligned}$$

Example:

$$\begin{aligned}
&\lim_{h\to 0}\frac{(2+h)^3-8}{h}=\frac{0}{0}\\
\hookrightarrow &\lim_{h\to 0}\frac{8+h^3+12h+6h^2-8}{h}=\lim_{h\to 0}\frac{h\left(h^2+6h+12\right)}{h}\\
&=0^2+6(0)+12\\
&=12
\end{aligned}$$

Example: $f(x)=\dfrac{x}{x}=1$ if $x\neq 0$

Example: $\lim_{x\to 1}\dfrac{3x-3}{x-1}=\dfrac{0}{0}\to$ undetermined.[1]

$$\lim_{x\to 1}\frac{3x-3}{x-1}=\lim_{x\to 1}\frac{3(x-1)}{(x-1)}=\lim_{x\to 1}3=3$$
$$x-1\neq 0$$

Example: $\lim_{x\to 1}\dfrac{x^2+2x+1}{x^4-1}=\dfrac{4}{0}=+\infty$

b) Rationalize

Example:

$$\begin{aligned}
&\lim_{x\to 0}\frac{\sqrt{x^2+9}-3}{x^2}=\frac{0}{0}\to\text{(indeterminate)}\\
&\to\lim_{x\to 0}\frac{\left(\sqrt{x^2+9}-3\right)\left(\sqrt{x^2+9}+3\right)}{x^2\left(\sqrt{x^2+9}+3\right)}\\
&=\lim_{x\to 0}\frac{x^2+x-9}{x^2\left(\sqrt{x^2+9}+3\right)}\\
&=\lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{x^2+9}+3\right)}\\
&=\lim_{x\to 0}\frac{1}{\sqrt{x^2+9}+3}\\
&=\frac{1}{6}
\end{aligned}$$

[1] Evil factor $x-1$ needs to get rid of.

Example:

$$\begin{aligned}
& \lim_{x\to 0.5^-} \frac{2x-1}{|2x^3-x^2|} \\
=& \lim_{x\to 0.5^-} \frac{2x-1}{|x^2(2x-1)|} \\
=& \lim_{x\to 0.5^-} \frac{2x-1}{x^2|2x-1|} \\
=& \lim_{x\to 0.5^-} \frac{2x-1}{x^2[-(2x-1)]} \\
=& \lim_{x\to 0.5^-} \frac{(2x-1)}{-x^2(2x-1)} \\
=& \lim_{x\to 0.5^-} -\frac{1}{x^2} \\
=& -4
\end{aligned}$$

Tip

$\lim_{x\to a} f(x) = L$ means when x is very close to a, then y-value is infinitely close to L

What are the possibilities when we do limits?

$$\lim_{x\to a} f(x) = \begin{cases} \text{number ok} \\ \dfrac{\text{number}}{0} \end{cases}$$

Example:

$$\begin{aligned}
\lim_{x\to 1^+} \frac{-4}{x-1} &= -\infty \quad \because \quad x-1 \to \text{small } \oplus \\
\lim_{x\to 1^-} \frac{-4}{x-1} &= +\infty \quad \because \quad x-1 \to \text{small } \ominus
\end{aligned}$$

Since,

$$\begin{aligned}
& \lim_{x\to 1^+} \left(\frac{-4}{x-1}\right) \neq \lim_{x\to 1^-} \left(\frac{-4}{x-1}\right) \\
\Rightarrow & \lim_{x\to 1} \left(\frac{-4}{x-1}\right) = \text{ DNE}
\end{aligned}$$

Tip

Generally, if

$$\lim_{x\to a^+} f(x) \neq \lim_{x\to \bar{a}} f(x)$$

then $\lim_{x\to a} f(x)$ does not exist (DNE)

RULE 2: Indeterminate form, $\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, \cdots$

Techniques:

1. Divide out

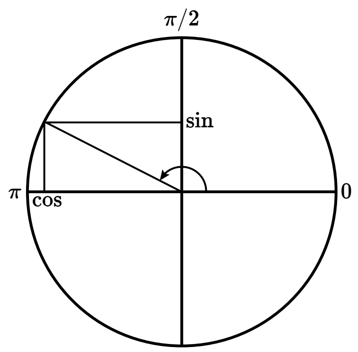
2. Rationalize
3. Rewrite
4. L'hopitals rule $\to$ later

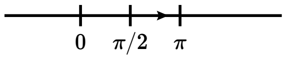
Example:

$$\lim_{x\to -\bar{3}} \frac{x+2}{x+3} = \frac{-1}{-\epsilon} = +\infty$$

Example:

$$\lim_{x\to \bar{\pi}} \cot x = \lim_{x\to \bar{\pi}} \frac{\cos x}{\sin x} = \frac{-1}{\text{small} \oplus} = -\infty$$





Example:

$$\lim_{x\to \bar{2}} \frac{x^2-2x}{x^2-4x+4} = \frac{0}{0} \to \text{indeterminate}$$

$$\hookrightarrow \text{Evil factor: } x-2$$

$$\begin{aligned} &\lim_{x\to \bar{2}} \frac{x^2-2x}{x^2-4x+4} \\ &= \lim_{x\to 2^-} \frac{x(x-2)}{(x-2)(x-2)} \\ &= \lim_{x\to \bar{2}^-} \frac{x}{x-2} \\ &= \frac{2}{\text{small} \ominus} \\ &= -\infty \end{aligned}$$

Example:

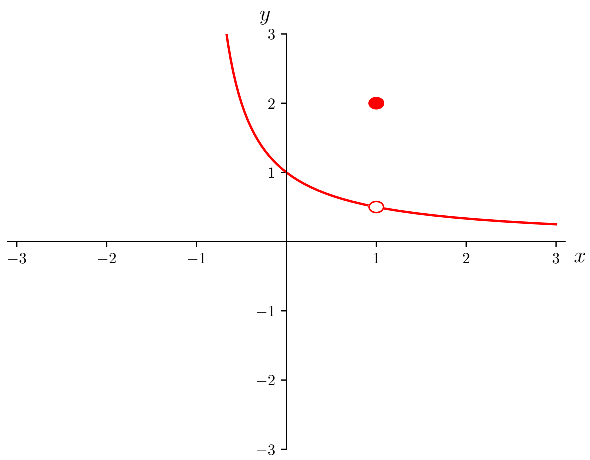
$$f(x) = \begin{cases} \dfrac{x-1}{x^2-1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

(a) What is $f(1)$? 2

(b) What is $\lim_{x\to 1} f(x)$? $\lim_{x\to 1} \frac{x-1}{x^2-1} = \frac{0}{0}$ (indeterminate)

$$
\begin{aligned}
&= \lim_{x\to 1} \frac{(x-1)}{(x-1)(x+1)} \\
&= \lim_{x\to 1} \frac{1}{x+1} \\
&= \frac{1}{2}
\end{aligned}
$$

Note: here, the limit did not split into 1^- and 1^+because it is the same.



$f(x) = [[x]]$: largest integer $\leq x$

Example:

$$
\begin{aligned}
&[[4]] = 4 \\
&[[4 \cdot 1]] = 4 \\
&[[4 \cdot 999]] = -4 \\
&[[\sqrt{2}]] = 1 \\
&[[-1/2]] = -1
\end{aligned}
$$

Example:

$$
\begin{aligned}
&\lim_{x\to 3^+} [[x]] = 3 \\
&\lim_{x\to 3^-} [[x]] = 2 \\
\Rightarrow &\lim_{x\to 3^-} [[x]] = \text{ DNE}
\end{aligned}
$$

Example:

$$
\begin{aligned}
&\lim_{x\to 2\cdot 1^+} [[x]] = 2 \\
&\lim_{x\to 2\cdot 1^-} [[x]] = 2 \\
&\lim_{x\to 2\cdot 1} [[x]] = 2
\end{aligned}
$$

Example:

$$\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t^2+t}\right)=\infty-\infty \text{ (indeterminate)}$$

Rewrite as:

$$\begin{aligned}
&\lim_{t\to 0}\left(\frac{1}{t}-\frac{1}{t(t+1)}\right)\\
=&\lim_{t\to 0}\left(\frac{t+1-1}{t(t+1)}\right)\\
=&\lim_{t\to 0}\left(\frac{t}{t(t+1)}\right)\\
=&\lim_{t\to 0}\left(\frac{1}{t+1}\right)\\
=&1
\end{aligned}$$

Example:

$$\frac{\sin(0.0001)}{(0.0001)^3}$$
$$\lim_{x\to 0}\frac{\sin(x)}{x^3}$$
$$\lim_{x\to 0}\frac{\sin\left(x^2\right)}{x^2}=1$$

Example:

$$\lim_{x\to 0}\frac{\sin\left(x^3\right)}{4x^5}=?$$

Example:

$$\begin{aligned}
&\lim_{x\to 0}\frac{\sin\left(x^2\right)\cos\left(x^3\right)}{x}=\frac{0}{0}\\
=&\lim_{x\to 0}\frac{x\cdot\sin\left(x^2\right)}{x^2}\cdot\cos\left(x^3\right)\\
=&\lim_{x\to 0}\frac{\sin\left(x^2\right)}{x^2}\cdot x\cdot\cos\left(x^3\right)\\
=&1\cdot 0 \quad \because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1\\
=&0
\end{aligned}$$

Example:

$$\begin{aligned}
&\lim_{x\to 0}\frac{\sin\left(4x^3\right)}{x^3}\\
=&\lim_{x\to 0}\frac{4\sin\left(4x^3\right)}{4x^3}\\
=&4\cdot 1\\
=&4
\end{aligned}$$

Example:

$$\lim_{x\to 0} \frac{\sin(x)}{x^2} = \textbf{DNE}$$

$$\lim_{x\to 0^+} \frac{\sin x}{x} \cdot \frac{1}{x} \mid \lim_{x\to 0^-} \frac{\sin x}{x} \cdot \frac{1}{x}$$

$$+\infty \mid -\infty$$

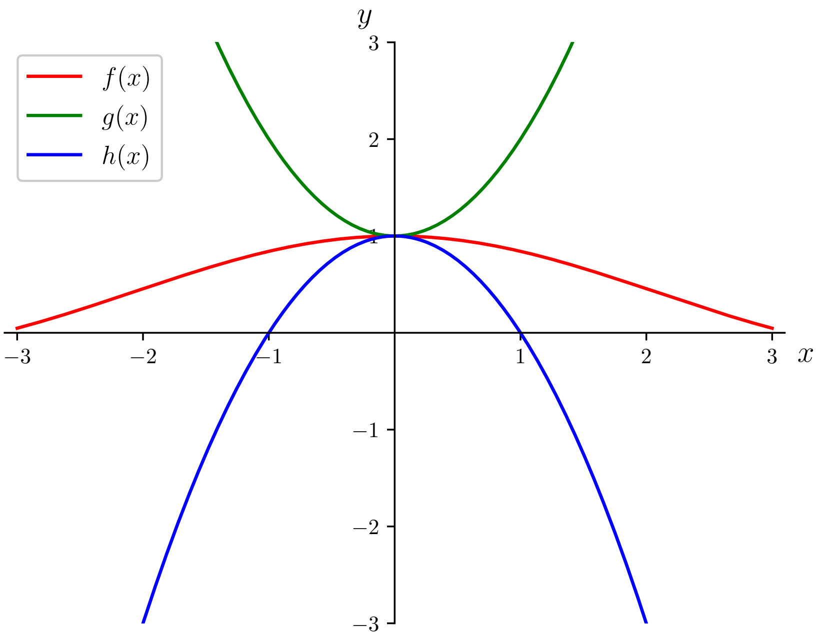
Example:

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0 \cdot (-1 \leq \textbf{no.} \leq 1) = 0$$

- cannot use $\lim_{x\to 0} \frac{\sin x}{x} = 1$

$$\because \lim_{x\to 0} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{y\to\infty} \frac{\sin y}{y} \neq 1$$

2.1.1 Squeeze theorem



$\lim_{x\to a} f(x)$: Cannot find limit but can find two other functions such that $h(x) \leq f(x) \leq g(x)$. and if $\lim_{x\to a} h(x) = \lim_{x\to a} g(x) = L$ then

$$\lim_{x\to a} f(x) = L$$

Example: $\lim_{x\to 0} x \sin\left(\frac{1}{x}\right)$

we know that

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

$$\Rightarrow \quad -x \leq x \sin\left(\frac{1}{x}\right) \leq x$$

$$\Rightarrow h(x) = -x \quad ; \quad g(x) = x$$

$$\lim_{x \to 0^{+-}} h(x) = \lim_{x \to 0^{+-}} (-x) = 0$$
$$\lim_{x \to 0^{+-}} g(x) = \lim_{x \to 0^{+1-}} (x) = 0$$

$\Rightarrow \lim_{x \to 0^{+}-} x \sin\left(\frac{1}{x}\right) = 0$ by squeeze theorem.

Example:

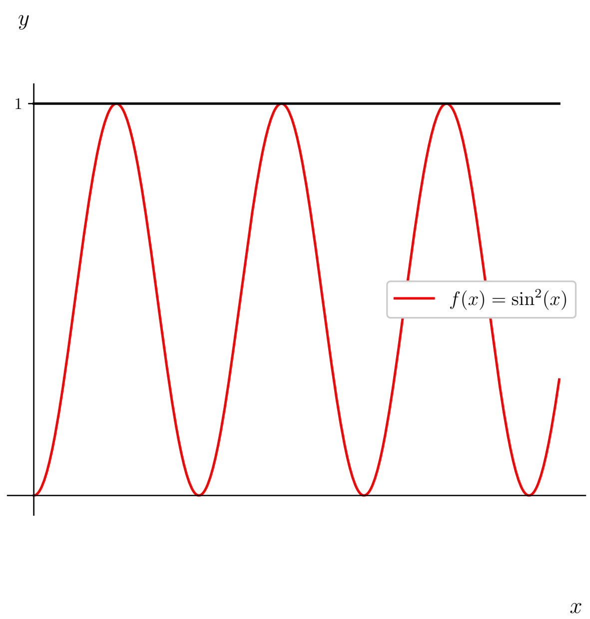
$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\sin\theta} = \lim_{\theta \to 0} \frac{\frac{\cos\theta - 1}{\theta}}{\frac{\sin\theta}{\theta}} = \frac{0}{1} = 0$$

Example:

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\sin\theta} \cdot \frac{\cos\theta + 1}{\cos\theta + 1}$$
$$= \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\sin\theta(\cos\theta + 1)}$$
$$= \lim_{\theta \to 0} \frac{-\sin^2\theta}{\sin\theta(\cos\theta + 1)}$$
$$= \lim_{\theta \to 0} \frac{-\sin\theta}{\cos\theta + 1}$$
$$= \frac{-0}{1 + 1}$$
$$= 0$$

Example:

$$\lim_{x \to 0^{+}} \sqrt{x}\left(1 + \sin^2\left(\frac{2\pi}{x}\right)\right)$$



We know that

$$
\begin{aligned}
-1 \leq \sin\left(\frac{2\pi}{x}\right) &\leq 1 \\
0 \leq \sin^2\left(\frac{2\pi}{x}\right) &\leq 1 \\
1 \leq 1+\sin^2\left(\frac{2\pi}{x}\right) &\leq 2 \\
\sqrt{x} \leq \sqrt{x}\left(1+\sin^2\left(\frac{2\pi}{x}\right)\right) &\leq 2\sqrt{x}
\end{aligned}
$$

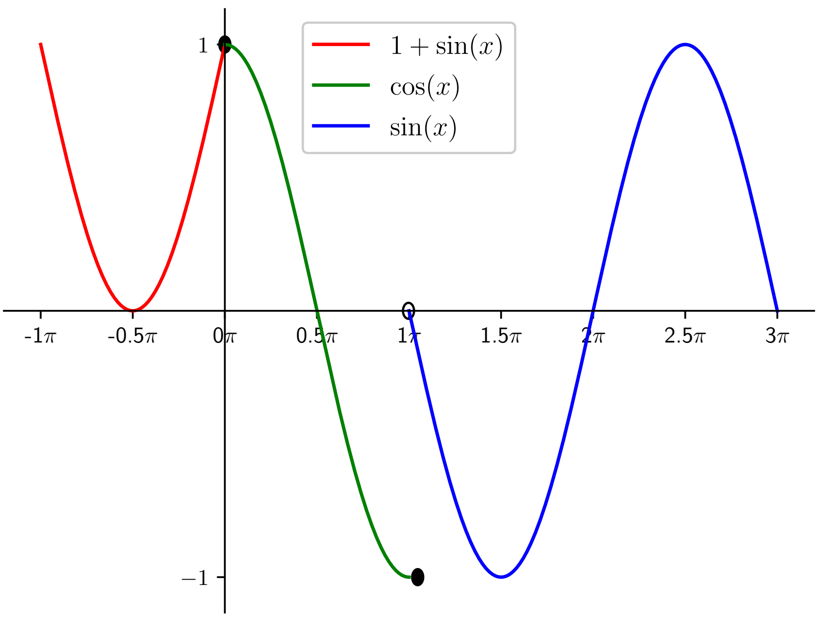
Since $\lim_{x\to 0^+} \sqrt{x} = \lim_{x\to 0^+} 2\sqrt{x} = 0.$

Thus $\lim_{x\to 0^+} \sqrt{x}\left(1+\sin^2\left(\frac{2\pi}{x}\right)\right) = 0$

Example:

$$
f(x) = \begin{cases} 1+\sin x & x<0 \\ \cos x & 0 \leq x \leq \pi \\ \sin x & x > \pi \end{cases}
$$

$\lim_{x\to a} f(x)$. What a?



$\therefore R1\{\pi\}$ limit exists for all a except when $a=\pi$.

Example: $\lim_{x\to -2} \frac{x+2}{x^3+8} = \frac{0}{0}$ (indeterminat).

* Evil factor: $(x+2)$

$$
\begin{aligned}
&\lim_{x\to -2} \frac{(x+2)}{(x+2)(x^2-2x+4)} \\
&= \lim_{x\to -2} \frac{1}{x^2-2x+4} = \frac{1}{12}
\end{aligned}
$$

2.1.2 Summary

- Limits.
- How to calculate? Plug in.

  $\rightarrow$ number (also includes $0/1 = 0$)

  $\rightarrow$ number $/0 \rightarrow +\infty/-\infty$

  $\rightarrow$ Indeterminate Form $\left(\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0.\infty, \cdots\right)$

  $\hookrightarrow$ Divide out

  $\hookrightarrow$ Rationalize

  $\hookrightarrow$ Manipulate (when there is also absolute value).

  $\rightarrow$ Squeeze theorem

  $\rightarrow$ L'hopital's Rule.

- Limits to remember.

  1. $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$
  2. $\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0$